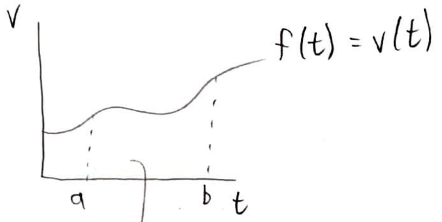
Lecture 27

Recall from Calc 2 the FDTC

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$



$$D = \int_a^b v(t)\,dt = x(b) - x(a)$$

This can be extended to line integrals.

Fundamental Theorem of Line Integrals

We've already discussed how the gradient is in a sense the derivative of a function of multiple variables. So it is perhaps no surprise that,

$$\int_C \nabla f \cdot d\bar{r} = f(\bar{r}(b)) - f(\bar{r}(a)).$$

This holds if C is a smooth curve parameterized by $\bar{r}(t)$ with $t \in [a,b]$ and f is differentiable whose gradient vector, ∇f, is continuous on C.

Ex. 1 Let C be the curve from $(1,-1,-\frac{1}{2})$ to $(1,1,\frac{1}{2})$ parameterized by $\bar{r}(t) = \langle -\cos(\pi t^4), t^{5/3}, \frac{t}{t^2+1} \rangle$ for $-1 \le t \le 1$ and let $\bar{F}(x,y,z) = \langle 2xy + z^2, x^2, 2xz + \pi\cos(\pi z) \rangle$. Find $\int_C \bar{F} \cdot d\bar{r}$.

We need to find f s.t. $\bar{F} = \nabla f$.

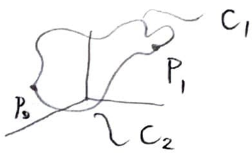
$$f(x,y,z) = x^2 y + xz^2 + \sin(\pi z)$$

$$\int_C \bar{F} \cdot d\bar{r} = f(1,1,\tfrac{1}{2}) - f(1,-1,-\tfrac{1}{2}) = (1 + \tfrac{1}{4} + 1) - (-1 + \tfrac{1}{4} - 1) = 4$$

Independence of Path

The Fundamental Theorem of Line Integrals says that $\int_C \nabla f \cdot d\bar{r}$ depends only on the value of f at the initial and terminal points. Thus if there are two curves C_1 + C_2 with the same initial and terminal points but otherwise differ, then,

$$\int_{C_1} \nabla f \cdot d\bar{r} = \int_{C_2} \nabla f \cdot d\bar{r}.$$



For this reason we say $\int_C \nabla f \cdot d\bar{r}$ is independent of path.

Conservation of Energy

Newton's Second law states $\bar{F}(\bar{r}(t)) = m\bar{a}(t) = m r''(t)$. The work done by a force between $\bar{r}(a)$ and $\bar{r}(b)$ is,

$$W = \int_C \bar{F} \cdot d\bar{r} = \int_a^b \bar{F}(\bar{r}(t)) \cdot \bar{r}'(t)\,dt = \int_a^b m\bar{r}''(t) \cdot \bar{r}'(t)\,dt$$

$$= \frac{m}{2} \int_a^b \frac{d}{dt}\left[r'(t) \cdot r'(t)\right] dt = \frac{m}{2} \int_a^b \frac{d}{dt} \|r'(t)\|^2\,dt$$

$$W = \frac{m}{2}\|r'(b)\|^2 - \frac{m}{2}\|\bar{r}'(a)\|^2 = \frac{1}{2}m v_b^2 - \frac{1}{2} m v_a^2 = K_B - K_A$$

If the force $\bar{F}$ is conservative and $\bar{F} = -\nabla P$

$$W = \int_C \bar{F} \cdot d\bar{r} = -\int_C \nabla P \cdot d\bar{r} = -\left[P(r(b)) - P(r(a))\right] = P_B - P_A$$

$$K_B - K_A = P_B - P_A \longrightarrow K_A + P_A = K_B + P_B$$

This is conservation of Energy!

Ex. 2 If $\bar{F}(x,y) = \langle 3 + 2xy,\ x^2 - 3y^2 \rangle$, find f s.t. $\bar{F} = \nabla f$. Evaluate $\int_C \bar{F} \cdot d\bar{r}$ where C is given by

$$\bar{r}(t) = \langle e^t \sin(t),\ e^t \cos(t) \rangle \qquad 0 \le t \le \pi$$

$f_x(x,y) = 3 + 2xy \rightarrow f(x,y) = 3x + x^2y + g(y)$

$f_y(x,y) = x^2 - 3y^2$

$f_y(x,y) = x^2 + g'(y)$

$g'(y) = -3y^2$

$g(y) = -y^3 + C$

$f(x,y) = 3x + x^2y - y^3 + C$

$r(\emptyset) = (\emptyset, 1) \quad r(\pi) = (\emptyset, -e^{\pi})$

$\int_C \bar{F} \cdot d\bar{r} = \int_C \nabla f \cdot d\bar{r} = f(\emptyset, -e^{\pi}) - f(\emptyset, 1) = e^{3\pi} + 1$